Vanishing theorems for products of exterior and symmetric powers

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1 Introduction

Many problems of algebraic geometry involve vanishing theorems for cohomology groups of ample vector bundles.

For ample vector bundles E over compact complex varieties X and a Schur functor S_I corresponding to an arbitrary partition I of the integer $|I|$, one would like to know the optimal vanishing theorem for the cohomology groups $H^{p,q}(X, S_I(E))$, depending on the rank of E and the dimension n of X. For $p = n$, a conjectural vanishing statement has been motivated by the study of the connectedness of degeneracy loci [La].

In an unpublished paper one of us proved a vanishing theorem for the situation where the partition I is a hook [Na]. Here we give a simpler proof of this theorem. In Theorem 2.2 we treat the same problem under weaker positivity assumptions, in particular under the hypothesis of ample $\Lambda^m E$ with $m \in \mathbf{N}^*$. In this case we also need some bound on the weight $|I|$ of the partition. Moreover, we prove that the same vanishing condition applies for $H^{q,p}(X, S_I(E))$, with p, q interchanged.

2 Statement and Results

Let E a vector bundle of rank e. For $0 \leq \alpha < k$ the hook Schur functor $\Gamma_k^\alpha E$, corresponding to the partition $(\alpha+1, 1, \ldots, 1)$ of $k \in \mathbf{N}^*$, can be defined inductively as follows:

$$\Gamma_k^0 E = \Lambda^k E$$

and

$$\Lambda^{k-\alpha} E \otimes S^\alpha E = \Gamma_k^{\alpha-1} E \oplus \Gamma_k^\alpha E$$

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for $0 < \alpha < k$. In particular, $\Gamma_k^{k-1}E = S^kE$. Note that $\Gamma_k^\alpha E = 0$ for $e - k + \alpha < 0$.

Define a function $\delta : \mathbf{N} \to \mathbf{N}^*$ by:

$$\binom{\delta(x)}{2} \leq x < \binom{\delta(x)+1}{2} .$$

In particular, $\delta(\binom{m}{2}) = m$ and $\delta(0) = 1$.

Theorem 2.1 (Nahm 1995) **:** *Let X be a compact complex variety of dimension n and E an ample vector bundle over X of rank e. Then,*

$$H^{p,q}(X, \Gamma_k^\alpha E) = 0,$$

for $q + p - n > (\delta(n-p) + \alpha)(e - k + 2\alpha) - \alpha(\alpha + 1)$.

For $\alpha = k - 1$ one obtains $H^{p,q}(X, S^\alpha E \otimes detE) = 0$, when $q + n - p > (\delta(n-p) - 1)k$. For $p = n$, this specialises to Griffith's vanishing theorem [Gr].

For $\alpha = 0$ one obtains $H^{p,q}(X, \Lambda^k E) = 0$ for $q + p - n > \delta(n-p)(e-k)$. For $p = n$, this specialises to Le Potier's vanishing theorem [LeP].

Here we prove the more general

Theorem 2.2 : *Let X be a compact complex variety of dimension n. Let $k = ml + s$ with $k, m, l \in \mathbf{N}^*$, $0 \leq s < m$. Let E be a vector bundle over X of rank e, such that $S_{I(l,m,s)}E$ is ample, with the partition of k*

$$I(l,m,s) = (\underbrace{l+1, \ldots, l+1}_{s \ times} , \underbrace{l, \ldots, l}_{(m-s) \ times}) .$$

Let $0 \leq \alpha < k$, $0 < p \leq n$, and

$$r = \delta\left(n - p + \binom{m}{2}\right) .$$

Suppose that one of the following four conditions is true:

– $m = 1$

– $\alpha = 0$, $(r-1)k > r(n-p-1)$

– $\alpha \geq 1$, $(r + \alpha - 2)k > (r + \alpha - 1)\left(n - p + ra + \binom{\alpha}{2} - 1\right) + \delta_{\alpha,2}\delta_{m,2}\delta_{n-p,1}$
(with Kronecker's δ-notation)

– $k > e + 1$, $(r + \alpha - \beta - 1)k > (r + \alpha - \beta)\left(n - p + ra + \binom{\alpha}{2} - \binom{\beta}{2} - 1\right)$,
where $\beta = k - e$.

Then

$$H^{p,q}(X, \Gamma_k^\alpha E) = H^{q,p}(X, \Gamma_k^\alpha E) = 0,$$

when

$$q + p - n > (r + \alpha)(e - k + \alpha) + \alpha(r - 1) \ .$$

Remark 2.3 : For $m = 1$, the condition $S^l E$ ample is equivalent to E ample, such that we obtain theorem 2.1 with the additional freedom to interchange p and q. When k is divisible by m, the ampleness condition is equivalent to $\Lambda^m E$ ample, as we shall show now. At the same time, we shall see that theorem 2.2 only yields vanishing statements for ample vector bundles, as expected.

Recall the dominance partial order of partitions $I = (i_1, i_2 \ldots)$ of $|I|$ and $J = (j_1, j_2 \ldots)$ of $|J|$. Usually it is defined under the assumption that I, J are partitions of the same integer, but we generalise it to arbitrary pairs of partitions by scaling I and J to rational partitions of 1. Let $\bar{i}_k = i_k/|I|$, $\bar{j}_k = j_k/|J|$. We say that S_I dominates S_J, if $\Sigma_{k=1}^l \bar{i}_k \geq \Sigma_{k=1}^l \bar{j}_k$ for any l. If each one dominates the other, i.e. if I and J are proportional, we call these Schur functors equivalent. For example, $S_{I(l,m,s)}$ dominates Γ_k^α and $S_{I(l,m,0)}$ is equivalent to Λ^m.

Lemma 2.4 : *If S_I dominates S_J, then ampleness of $S_I E$ implies ampleness of $S_J E$.*

Proof: It is sufficient to show that every irreducible direct summand of $S_J E^{\otimes k|I|}$ is isomorphic to a direct summand of $S_I E^{\otimes k|J|}$ for some $k > 0$. More generally, consider the set T' of partitions I' with $|I'| = |J|k|I|$ which are dominated by $k|J|I$. It is sufficient to show that for any $I' \in T'$ the bundle $S_{I'} E$ is isomorphic to a direct summand of $S_I E^{\otimes k|J|}$ when k is a (sufficiently large) multiple of $lcm(1, 2, \ldots, e)$. In this case, all vertices of the convex hull of T' in $\mathbf{R}^e$ different from $(|J|k|I|/e, \ldots, |J|k|I|/e)$ lie on a face F_l passing through $k|J|I$. Such a face is given by the equation $\Sigma_{n=1}^l \bar{i}_n = \Sigma_{n=1}^l \bar{i'}_n$ for some $l < e$.

For partitions I of length at most e let $V(I) = S_I \mathbf{C}^e$ be the corresponding highest weight representation of $GL_e(\mathbf{C})$. Define

$$T_{e,r} = \{(I_1, \ldots, I_r, I') \mid V(I') \subset V(I_1) \otimes \ldots \otimes V(I_r)\} \ .$$

Recently, Knutson and Tao have shown that the semigroups $T_{e,r}$ are saturated, such that for fixed $I_1, \ldots, I_r$ the set $\{I' | (I_1, \ldots, I_r, I') \in T_{e,r}\}$ is convex. More precisely, it contains all integral points of its convex hull H in $\mathbf{R}^e$ [K-T] (see also [Bu] and [Ze]).

Let us specialize to $r = k|J|$ and $I_1 = \ldots = I_r = I$. It remains to show that the vertices of the convex hull of T' lie in H. For the vertices on F_l this question reduces to the analogous problem for $GL_l(\mathbf{C})$ and $GL_{e-l}(\mathbf{C})$. Thus by induction in e it is sufficient to consider the vertex $(|J|k|I|/e, \ldots, |J|k|I|/e)$. In other words, does $V(I)^{\otimes ke}$ contain a one-dimensional representation of $GL_e(\mathbf{C})$, when

k is sufficiently large? By explicit application of the Littlewood-Richardson rule one can show that already $k = 1$ is sufficient. It is more convenient, however, to consider large k and the standard character formula for the multiplicity of the trivial representation of $SU(e)$ in $V(I)^{\otimes ke}$. When χ is the character of the $SU(e)$ representation $V(I)$, this multiplicity is given by the integral of χ^{ke} over the Haar measure of $SU(e)$. With increasing k the multiplicity becomes arbitrarily large, since the dominant contributions come from small neighborhoods of the central elements of $SU(e)$. $\square$

Corollary 2.5 : *In the notation of theorem 2.2, ampleness of $S_{I(l,m,0)}E$ and $\Lambda^m E$ are equivalent. Ampleness of $S_{I(l,m,s)}E$ implies ampleness of $\Gamma_k^\alpha E$.*

Before we come to the proof of theorem 2.2, let us prove a proposition which allows us to get rid of any lower bound on k when $m = 1$. The procedure may be useful in more general situations, so we will be somewhat more general than necessary in our context.

Let us recall that a line bundle L on a variety X is nef if $(C.L) \geq 0$ for every curve C in X and a vector bundle F is nef, when the line bundle $\mathcal{O}_{P(F)}(1)$ is nef on $\mathbf{P}(F)$. If F is nef then $S_I F$ is nef for any partition I.

Bloch-Gieseker Lemma [B-G, Lemma.2.1] *Let L be a line bundle on a projective variety X and d a positive integer. Then there exist a projective variety Y, a finite surjective morphism $f : Y \to X$, and a line bundle M on Y such that $f^*L \simeq M^d$.*

Lemma 2.6 : *Let X be a projective variety. Let E and F_i with $i = 1, \ldots, m$ be vector bundles on X of ranks e, f_i with E ample and the F_i nef. Let I, J_i be non-vanishing partitions. Then there exist a projective variety Y, a finite surjective morphism $f : Y \to X$, and ample vector bundles E', F_i' on Y of ranks e, f_i such that $f^*(S_I E \otimes_{i=1}^m S_{J_i} F_i) = S_I E' \otimes S_{J_i} F_i'$.*

Proof: If E is ample, $S^k E \otimes (detE)^*$ is ample for $k >> 0$. We fix k accordingly and use the B-G lemma such that $f_\alpha^*(detE) = M^{k\alpha}$. Then $E'_\alpha = f_\alpha^* E \otimes (M^*)^\alpha$ is ample since $S^k E'_\alpha$ is. For $\alpha = \sum_i |J_i|$ we have $f_\alpha^*(S_I E \otimes_{i=1}^m S_{J_i} F_i) = S_I E'_\alpha \otimes S_{J_i} F_i'$ where $F_i' = M^{|I|} \otimes f_\alpha^* F_i$. According to Fujita's lemma [Fu], the F_i' are ample, too. $\square$

Corollary 2.7 : *To prove a vanishing theorem $H^{p,q}(X, S_I E \otimes_{i=1}^m S_{J_i} F_i) = 0$ for fixed p, q, n, e, f_i under the assumption E is ample and the F_i are nef, it is sufficient to treat the case when E and the F_i are all ample.*

Indeed, the vanishing of $H^{p,q}(Y, f^*\mathcal{F})$ implies the vanishing of $H^{p,q}(X, \mathcal{F})$ for any vector bundle $\mathcal{F}$ on X and any finite surjective morphism $f : Y \to X$. $\square$

Now we can show how to get rid of lower bounds on k when $m = 1$.

Proposition 2.8 : *Fix $n, p, q, \alpha \in \mathbf{N}$ and $t \in \mathbf{Z}$. Assume that $H^{p,q}(X, \Gamma_k^\alpha E)$ vanishes for some k, all compact varieties X of dimension n and all ample vector bundles E of rank $e = k + t$. Let $k' < k$. Then $H^{p,q}(X, \Gamma_{k'}^\alpha E')$ vanishes for all ample vector bundles E' of rank $e' = k' + t$.*

Proof: For given E', put $E = E' \oplus L^{\oplus(k-k')}$, where L is any ample line bundle. Since $\Gamma_{k'}^\alpha E' \otimes L^{k-k'}$ is a direct summand of $\Gamma_k^\alpha E$, we have

$$H^{p,q}(X, \Gamma_{k'}^\alpha E' \otimes L^{k-k'}) = 0$$

for ample vector bundles E' of rank e' and ample line bundles L. By Corollary 2.7, this vanishing result remains true, when L is replaced by the trivial line bundle. □

3 Proof of the theorem 2.2.

To prepare the proof, we need a lemma and the definition of the Borel-Le Potier spectral sequence, which has been made a standard tool in the derivation of vanishing theorems [Dem].

Let E be a vector bundle over a compact complex variety X. Considering subspaces of codimension $s_1, s_2, \ldots$ in the fibres, one obtains the corresponding incomplete flag manifold $Y = M_{s_1,s_2,\ldots}(E)$, with a natural projection $\pi : Y \to X$. Let Q_k with $rank(Q_1) = s_1$, $rank(Q_2) = s_2 - s_1, \ldots$ be the corresponding canonical quotient bundles over Y. Define the line bundles

$$Q^{i_1,i_2,\cdots} = \bigotimes_k det(Q_k)^{i_k} \; .$$

Lemma 3.1 : *Let I be a partition of the form*

$$I = (\underbrace{i_1, \ldots, i_1}_{s_1 \; times}, \; \underbrace{i_2, \ldots, i_2}_{(s_2 - s_1) \; times}, \ldots)$$

such that $S_I E$ is ample over X. Then $Q^{i_1,i_2,\cdots}$ is ample over Y.

Proof: See [Dem], Lemmata 2.11, 4.1. □

The projection π yields a filtration of the bundle Ω_Y^P of exterior differential forms of degree P, namely

$$F^p(\Omega_Y^P) = \pi^* \Omega_X^P \wedge \Omega_Y^{P-p}.$$

The corresponding graded bundle is given by

$$F^p(\Omega_Y^P)/F^{p+1}(\Omega_Y^P) = \pi^*\Omega_X^P \otimes \Omega_{Y/X}^{P-p},$$

where $\Omega_{Y/X}^{P-p}$ is the bundle of relative differential forms of degree $P-p$.

For a given line bundle $\mathcal{L}$ over Y the corresponding filtration of $\Omega_Y^P \otimes \mathcal{L}$ yields a spectral sequence which abuts on $H^{P,q}(Y,\mathcal{L})$. It has been named Borel-Le Potier spectral sequence by Demailly. It is given by the data $X, Y, \mathcal{L}, P$ and will be denoted by ${}^PE$. Its E_1-term

$${}^PE_1^{p,q-p} = H^q(Y, \pi^*(\Omega_X^p) \otimes \Omega_{Y/X}^{P-p} \otimes \mathcal{L})$$

can be calculated via the Leray spectral sequence.

Now let us come to the proof. For convenience, we add an index 0 to the variables l, s, α, p, q, r in theorem 2.2, such that without index they can be used as free variables in the proof. Thus we prove a vanishing theorem for $H^{p_0,q_0}(X, \Gamma_k^{\alpha_0}E)$ where $0 \le p_0, q_0 \le n$, $0 \le \alpha_0 < k$ and we define

$$r_0 = \delta\left(n - p_0 + \binom{m}{2}\right) .$$

We will work by induction in the finite set

$$B = \{(\alpha,p,q) \in \mathbf{N}^3 \mid k-e \le \alpha \le \alpha_0+r_0-m,\ p_0+Q(\alpha) \le p \le n,\ q_0+Q(\alpha) \le q \le n\},$$

where

$$Q(\alpha) = r_0(\alpha-\alpha_0) - \binom{|\alpha-\alpha_0|}{2} .$$

By a suitable ordering of B, we shall prove inductively that $H^{p,q}(X, \Gamma_k^{\alpha}E) = 0$ for all $(\alpha,p,q) \in B$, in particular for its maximal element (α_0, p_0, q_0).

For any $(\alpha,p,q) \in B$ consider the incomplete flag variety $Y = M_{s,r}(E)$ with $r = r_0 + \alpha_0 - \alpha$, $l = [k/r]$ and $s = k - lr$. This variety reduces to the Grassmannian $G_r(E)$ when $s = 0$. For the line bundle $\mathcal{L} = Q^{l+1,l}$ and

$$P = p - r\alpha + (l-1)\binom{r+1}{2} + rs - \binom{s}{2}$$

consider the corresponding Borel-Le Potier spectral sequence ${}^PE$. By the Leray spectral sequence,

$${}^PE_1^{p',q'-p'} = \bigoplus_{j\in\mathbf{N}} H^{p',q'-j}(X, R^j\pi_*(\Omega_{Y/X}^{P-p'} \otimes \mathcal{L})) .$$

We shall see that the r.h.s. can be evaluated in terms of hook Schur functors, if either $r = 1$ or $k \geq n - p + r\alpha + l$. For $m = 1$ we can assume that this lower bound is satisfied for all elements of B, since proposition 2.8 allows to get rid of it subsequently. Otherwise we have to show that $k > k(\alpha)$, where

$$k(\alpha) = (A(\alpha) - 1)(1 + (r_0 + \alpha_0 - \alpha - 1)^{-1})$$

and

$$A(\alpha) = n - p_0 - Q(\alpha) + r\alpha \ .$$

For $r_0 + \alpha_0 - \alpha = 1$ we put $k(\alpha) = -\infty$.

When $k(\alpha)$ is extended to real argument, it becomes a decreasing function of α in the interval $\alpha_0 \leq \alpha \leq r_0 + \alpha_0 - 1$, $1 \leq \alpha$, since

$$n - p_0 + r_0\alpha_0 + \binom{\alpha_0 + 1}{2} < \binom{r_0 + \alpha_0 + 1}{2} \ .$$

For $\alpha_0 = 0$, one checks easily that $k(0) \geq k(1)$, such that $k > k(0)$ implies $k > k(\alpha)$ for all α occuring in B.

Thus $k > k(0)$ implies $k > k(\alpha)$ for all α occuring in B, if $\alpha_0 = 0$. For $1 \leq \alpha \leq \alpha_0$ one has $k(1) \geq k(0)$. Moreover, the function $k(\alpha)$ is non-increasing for $\alpha \geq 2$, $r_0 + \alpha_0 \geq 6$ and for $\alpha \geq 1$, $r_0 + \alpha_0 \geq 7$. Thus the function $[k(\alpha)]$, α occuring in B, takes its maximum at $\alpha = 1$ or $\alpha = k - e$, except for a small number of cases for which the maximum of $[k(\alpha)]$ turns out to be $[k(1)]+1$. For $m > 1$ this only happens for $m = 2$, $n - p_0 = 1$, $\alpha_0 = 2$.

Let

$$\chi = (l - 1)\binom{r}{2} + rs - \binom{s + 1}{2} - (r - 1)\alpha \ .$$

Since either $r = 1$ or $k \geq n - p + r\alpha + l$ one has according to [Ma], Prop. 3 and Lemma 3,

$${}^{P}E_1^{p',q'+\chi-p'} = \bigoplus_{\beta=0}^{[(k-1)/r]} n_s(\sigma + p - p' + r(\beta - \alpha))\, H^{p',q'+p'-p+\alpha-\beta}(X, \Gamma_k^\beta E),$$

where $\sigma = rs - \binom{s}{2}$ and the multiplicity function n_s is generated by

$$\sum_{a,s} n_s(a) x^a z^s = \prod_{i=1}^{r} (1 + x^{r+1-i} z) \ .$$

Note that $n_s(a) = 0$ for $a > \sigma$ and $n_s(\sigma) = 1$.

Manivel states his result under the additional condition $e \geq k$, which is not necessary and not used in the proof. For $k > e$ the lower bound of the β summation can be replaced by $k - e$. The upper bound is given as l in Manivel's statement. This is correct for $s > 0$ but has to be reduced by 1 for $s = 0$, as is

clear from his proof. A priori, one only expects the occurence of Schur functors Γ_k^β which are dominated by $S_{I(l,r,s)}$, which indeed is true for the correct upper bound. Note that ${}^P E_1^{p,q+\chi-p}$ contains a direct summand $H^{p,q}(X, \Gamma_k^\alpha E)$.

In the Borel-Le Potier spectral sequence consider any non-vanishing morphism

$$ {}^P E_M^{p,q+\chi-p} \longrightarrow {}^P E_M^{p+M,q+\chi+1-p-M} , $$

with $M > 0$, or

$$ {}^P E_{-M}^{p+M,q+\chi-1-p-M} \longrightarrow {}^P E_{-M}^{p,q+\chi-p} , $$

with $M < 0$. Then ${}^P E_1^{p+M,q+\chi+sgn(M)-p-M}$ is a direct sum of terms $H^{p',q'}(X, \Gamma_k^\beta E)$ with $0 \le p', q' \le n$ such that $p' - p \ge r(\beta - \alpha)$ and

$$ q' - q = sgn(M) + \alpha - \beta + M \ge (r-1)(\beta - \alpha) + sgn(\beta - \alpha) . $$

From these conditions one obtains $(\beta, p', q') \in B$. Indeed, the conditions for p', q' follow from the easily verified relations $Q(\beta) - Q(\alpha) \le r(\beta - \alpha)$ and $Q(\beta) - Q(\alpha) \le (r-1)(\beta - \alpha) + sgn(\beta - \alpha)$. We still have to check that $\beta \le \alpha_0 + r_0 - m$. Note that we already have shown $p' \ge Q(\beta) + p_0$. By the definition of δ and Q one finds $Q(r_0 + \alpha_0 - m + 1) + p_0 > n$. Thus for $\beta = \alpha_0 + r_0 - m + 1$ and *a fortiori* for $\beta > \alpha_0 + r_0 - m + 1$ one has $p' > n$, which is excluded.

To prove the desired vanishing theorem by induction, we still need $(\beta, p', q') < (\alpha, p, q)$ in a suitable ordering. We use the variable

$$ L(\alpha, p) = 2(n-p) + \alpha(2(r_0 + \alpha_0) - \alpha) . $$

A short calculation yields $L(\beta, p') \le L(\alpha, p) - 1$ for any of the morphisms under consideration.

Assume that $H^{p',q'}(X, \Gamma_k^\beta E) = 0$ for all $(\beta, p', q') \in B$ with $L(\beta, p') < L(\alpha, p)$. Then

$$ {}^P E_1^{p,q+\chi-p} = {}^P E_\infty^{p,q+\chi-p} , $$

since the morphisms adjacent to ${}^P E_M^{p,q+\chi-p}$ all vanish. By the Kodaira-Akizuki-Nakano vanishing theorem the r.h.s. vanishes when $Q^{l+1,l}$ is ample and $P + q + \chi > dim(Y)$. By lemma 2.4 and lemma 3.1, the ampleness of $S_{I(l,m,s)}E$ guarantees indeed that $Q^{l+1,l}$ is ample for all $(\alpha, p, q) \in B$.

Since $dim(Y) = n + r(e-r) + s(r-s)$ and $p + q - n \ge p_0 + q_0 - n + 2Q(\alpha)$ the condition $P + q + \chi > dim(Y)$ reduces to

$$ p_0 + q_0 - n + \alpha(e - k + \alpha) + |\alpha - \alpha_0| + \alpha - \alpha_0 > (r_0 + \alpha_0)(e - k + \alpha_0) + \alpha_0(r_0 - 1) . $$

By assumption, p_0+q_0-n is strictly greater than the r.h.s. Since $\alpha(e-k+\alpha)$ and $|\alpha-\alpha_0|+\alpha-\alpha_0$ are both non-negative (they only vanish for the minimal value of α occuring in B), the conditions of the Kodaira-Akizuki-Nakano vanishing theorem are satisfied. Thus ${}^{P}E_1^{p,q+\chi-p}$ and its direct summand $H^{p,q}(X,\Gamma_k^\alpha E)$ do vanish. In particular, this is true for $(\alpha_0,p_0,q_0)\in B$, as was to be shown.

When we put

$$r_0=\delta\left(n-q_0+\binom{m}{2}\right)\ ,$$

one has to replace p by q in two passages of the proof. The first one occurs when one shows that $(\beta,p',q')\in B$. One finds $Q(r_0+\alpha_0-m+1)+q_0>n$. One now concludes for $\beta\geq\alpha_0+r_0-m+1$ that $q'>n$, which is excluded.

The second replacement concerns the induction variable which now becomes $L(\alpha,q)$. For any of the morphisms under consideration one finds $L(\beta,q')\leq L(\alpha,q)-1$. Apart from these single letter replacements, the proof goes through verbally. □

Remarks: The condition $k\geq k_0$ is necessary to exclude the occurence of non-hook partitions in the spectral sequence. It should be possible to weaken the lower bound by suitable vanishing theorems for these extra terms.

The ampleness condition of theorem 2.2 is not the only possible one. When E allows the extraction of a sufficiently positive line bundle, one can use the idea of proposition 2.8 to increase k to a value divisible by m.

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References.

- [B-G] S. Bloch, D. Gieseker, *The positivity of the Chern classes of an ample vector bundle,* Invent. Math. **12** (1971) 112-117.
- [Bu] A.S. Buch, *The Saturation Conjecture (after A. Knutson and T. Tao)*, preprint, November 1998.
- [Dem] J.P. Demailly, *Vanishing theorems for powers of an ample vector bundle,* Invent. Math. **91** (1988), 203-220.
- [Fu] T. Fujita, *Semipositive Line Bundles*, J. Fac. Sci. Univ. Tokyo Sect. IA Math. **30** (1983), 353-378.
- [Gr] P.A. Griffiths, *Hermitian diffferential geometry, Chern classes and positive vector bundles*, Global analysis, paper in honor of Kodaira (1969), 185-251.
- [K-T] A. Knutson and T. Tao *The honeycomb model of the Berenstein-Zelevinsky polytope I: Klyachko's saturation conjecture*, preprint, 1998.

- [La] F. Laytimi, *On Degeneracy Loci*, Int. J. Math. **7** (1996), 745-754.
- [LeP] J. Le Potier, *Annulation de la cohomology a valeurs dans un fibre vectoriel holomorphe positif de rang quelconque*, Math. Ann. **218** (1975), 35-53.
- [Ma] L. Manivel, *Un théorème d'annulation pour les puissances extérieures d'un fibré ample*, J. reine angew. Math. **422** (1991), 91-116.
- [Na] W. Nahm, *A vanishing theorem for products of exterior and symmetric powers*, Preprint. (1995), Univ. Bonn. Germany.
- [Ze] A. Zelevinsky, *Littlewood-Richardson semigroups*, MSRI Preprint, 1997-044.